Filter analysis and design

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Abstract

The filter design technique analysed in this tutorial is oriented toward the creation of causal linear-phase Finite Impulse Response (FIR) filters. It is based on the direct approximation of their desired frequency responses by windowing their corresponding impulse responses.

The desired frequency response of a given filter $H_d(e^{j\omega})$ (assumed ideal) is shown in Figure 1.

Note that the desired filter $H_d(e^{j\omega})$ has an ideal frequency-selective behaviour (it has step discontinuities) and also presents no time delay along the whole spectrum (its phase is a constant function that equals zero).

To approximate this desired frequency response (with unrestricted specifications), the corresponding impulse response is windowed (and thus truncated) to ensure that the filter has a finite impulse response. Hence, the first step is to calculate the impulse response of the filter $h_d[n]$ based on its desired frequency behaviour $H_d(e^{j\omega})$:

$$h_d[n] = \frac{G}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{G}{2\pi} \left( \int_{-\omega_c}^{\omega_c} e^{j\omega n} \, d\omega + \int_{\omega_c}^{\omega_c} e^{j\omega n} \, d\omega \right)$$

where the change of variable $u = j\omega n$ is applied,

$$h_d[n] = \frac{G}{2\pi jn} \left( e^{j\omega_c n} - e^{-j\omega_c n} + e^{-j\omega_c n} - e^{j\omega_c n} \right)$$
where

\[ \omega_{c1} < \omega_{c2} \]

\[ \omega_{c1}, \omega_{c2} \in [0, \pi] \]

Figure 1: Frequency response (magnitude and phase) of the desired/ideal filter \( H_d(e^{j\omega}) \).

where Euler’s formula is applied,

\[
\begin{align*}
    h_d[n] &= \frac{G}{\pi n} (\sin(\omega_{c2}n) - \sin(\omega_{c1}n)) \\
    &= \frac{G\omega_{c2}}{\pi} \text{sinc}(\omega_{c2}n) - \frac{G\omega_{c1}}{\pi} \text{sinc}(\omega_{c1}n)
\end{align*}
\]

Note that \( h_d[n] \) stretches to \( \infty \) (ideal frequency-selective response) and it is also defined for \( n < 0 \) (no delay, zero-phase system), see Figure 2. These impulse response characteristics result from unrestricted desired filter specifications. For practical purposes, a compromise has to be found between the features of the filter and its computational cost.

In a real-time application (where the samples are acquired in real time), Finite Impulse Response (FIR) filters must be causal\(^1\) and a generalised

\(^1\)Anticausal systems would only make sense when the whole amount of data to process was already recorded and available at the beginning of the filtering process.
linear-phase would be highly preferable in order to avoid phase distortion (a non-linear phase-response alters the temporal shape of the signal). Therefore, since zero-phase is not realisable for causal systems, some time delay must be allowed.

Given a filter $f[n]$, causality is attained by defining only a positive sequence

$$\exists f[n] \quad \forall n \geq 0$$  \hspace{1cm} (6)

and a generalised linear-phase is attained by defining a symmetry about an $\frac{M}{2}$ point

$$f[n] = f[M - n] \quad 0 \leq n \leq M$$  \hspace{1cm} (7)

In order to incorporate these requirements in the approximation $h[n]$ of the desired filter, its desired impulse response $h_d[n]$ is windowed by a finite-duration rectangular window $w[n]$ of $M + 1$ points through a multiplication

$$k[n] = h_d[n] w[n]$$  \hspace{1cm} (8)

where

$$w[n] = \begin{cases} 
1, & |n| \leq \frac{M}{2} \\
0, & otherwise 
\end{cases}$$  \hspace{1cm} (9)

and a time shift of $\frac{M}{2}$ samples

$$h[n] = k \left[ n - \frac{M}{2} \right]$$  \hspace{1cm} (10)

See how $h[n]$ looks like in Figure 3.

The approximated filter $h[n]$ has order $M$, and thus its impulse response has $M + 1$ points. For consistency with the notation used, $M$ must be an even integer so that $\frac{M}{2}$ is still an integer. Then, $h[n]$ has a symmetry about
Figure 3: Impulse response of the approximated filter $h[n]$. 

the $\frac{M}{2}$ point, which always coincides with one point of $h[n]$ (the central point of the sequence of $M + 1$ points).

In order to see how this windowing process affects the frequency response of the filter (compared to its desired behaviour) the Modulation or Windowing Theorem is of convenient use:

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) W(e^{j(\omega-\theta)}) \, d\theta$$  \hspace{1cm} (11)

Note that Eq. (11) is a periordic convolution, i.e. a convolution of two periodic functions with the limits of integration extending over only one period. Overall, $H(e^{j\omega})$ is a “smeared” version of the desired filter response $H_d(e^{j\omega})$. A graphical picture of this effect is shown in [Oppenheim and Schafer, 2009]. Bear in mind that the window function $W(e^{j\omega})$ does not need to be rectangular (in fact, the rectangular window is the simplest). More enhanced window functions yield better approximations, also see [Oppenheim and Schafer, 2009] for further details.

In the end, the obtained filter $H(e^{j\omega})$ has a frequency response like

$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{-j\omega \frac{M}{2}}$$  \hspace{1cm} (12)

In order to measure the linearity of the phase, the group function $\tau(\omega)$ represents a convenient measure:

$$\tau(\omega) = grd[H(e^{j\omega})] = -\frac{d}{d\omega} \angle H(e^{j\omega})$$  \hspace{1cm} (13)

Therefore, since $\tau(\omega) = \frac{M}{2}$, i.e. a constant delay for all frequencies, $H(e^{j\omega})$ is a linear-phase system and thus no dispersion in time of the output signal energy is present (no phase distortion).
References