Filter analysis and design

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Abstract

The filter design technique analysed in this tutorial is oriented toward the creation of causal linear-phase Finite Impulse Response (FIR) filters. It is based on the direct approximation of their desired frequency responses by windowing their corresponding impulse responses.

The desired frequency response of a given filter $H_d(e^{j\omega})$ (assumed ideal) is shown in Figure 1.

Note that the desired filter $H_d(e^{j\omega})$ has an ideal frequency-selective behaviour (it has step discontinuities) and also presents no time delay along the whole spectrum (its phase is a constant function that equals zero).

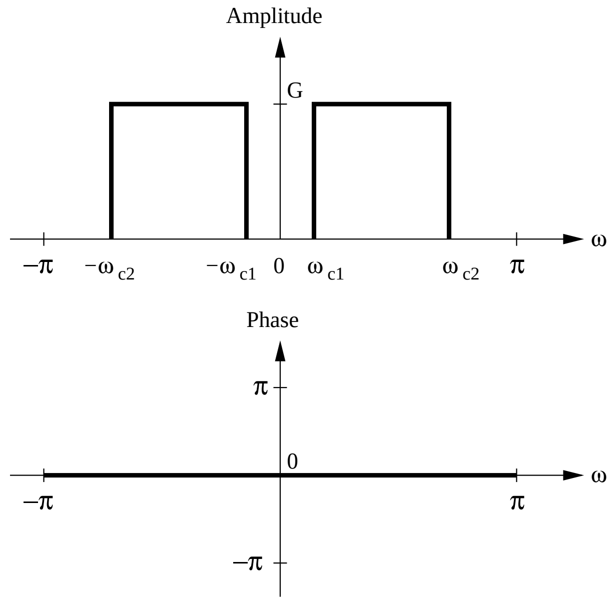
To approximate this desired frequency response (with unrestricted specifications), the corresponding impulse response is windowed (and thus truncated) to ensure that the filter has a finite impulse response. Hence, the first step is to calculate the impulse response of the filter $h_d[n]$ based on its desired frequency behaviour $H_d(e^{j\omega})$:

$$h_d[n] = \frac{G}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \quad (1)$$

$$= \frac{G}{2\pi}\left(\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega\right) \quad (2)$$

where the change of variable $u = j\omega n$ is applied,

$$h_d[n] = \frac{G}{2\pi jn}\left(e^{j\omega_{c2}n} - e^{-j\omega_{c2}n} + e^{-j\omega_{c1}n} - e^{j\omega_{c1}n}\right) \quad (3)$$



where

$$\omega_{c1} < \omega_{c2}$$

$$\omega_{c1}, \omega_{c2} \in [0, \pi]$$

Figure 1: Frequency response (magnitude and phase) of the desired/ideal filter $H_d(e^{j\omega})$.

where Euler's formula is applied,

$$h_d[n] = \frac{G}{\pi n} \left(\sin(\omega_{c2} n) - \sin(\omega_{c1} n)\right) \tag{4}$$

$$= \frac{G\omega_{c2}}{\pi} \operatorname{sinc}(\omega_{c2} n) - \frac{G\omega_{c1}}{\pi} \operatorname{sinc}(\omega_{c1} n) \tag{5}$$

Note that $h_d[n]$ stretches to ∞ (ideal frequency-selective response) and it is also defined for $n < 0$ (no delay, zero-phase system), see Figure 2. These impulse response characteristics result from unrestricted desired filter specifications. For practical purposes, a compromise has to be found between the features of the filter and its computational cost.

In a real-time application (where the samples are acquired in real time), Finite Impulse Response (FIR) filters must be causal[1] and a generalised

[1] Anticausal systems would only make sense when the whole amount of data to process was alredy recorded and available at the beginning of the filtering process.

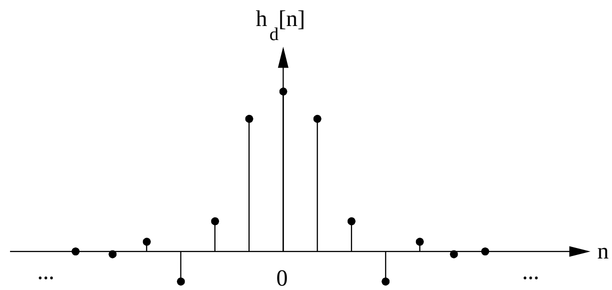


Figure 2: Impulse response of the desired/ideal filter $h_d[n]$.

linear-phase would be highly preferable in order to avoid phase distortion (a non-linear phase-response alters the temporal shape of the signal). Therefore, since zero-phase is not realisable for causal systems, some time delay must be allowed.

Given a filter $f[n]$, causality is attained by defining only a positive sequence

$$\exists f[n] \quad \forall n \geq 0 \tag{6}$$

and a generalised linear-phase is attained by defining a symmetry about an $\frac{M}{2}$ point

$$f[n] = f[M-n] \quad 0 \leq n \leq M \tag{7}$$

In order to incorporate these requirements in the approximation $h[n]$ of the desired filter, its desired impulse response $h_d[n]$ is windowed by a finite-duration rectangular window $w[n]$ of $M+1$ points through a multiplication

$$k[n] = h_d[n]\, w[n] \tag{8}$$

where

$$w[n] = \begin{cases} 1, & |n| \leq \frac{M}{2} \\ 0, & otherwise \end{cases} \tag{9}$$

and a time shift of $\frac{M}{2}$ samples

$$h[n] = k\left[n - \frac{M}{2}\right] \tag{10}$$

See how $h[n]$ looks like in Figure 3.

The approximated filter $h[n]$ has order M, and thus its impulse response has $M+1$ points. For consistency with the notation used, M must be an even integer so that $\frac{M}{2}$ is still an integer. Then, $h[n]$ has a symmetry about

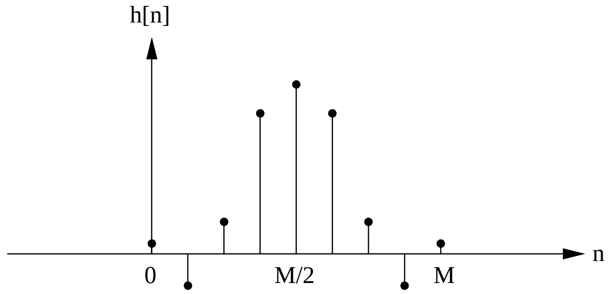


Figure 3: Impulse response of the approximated filter $h[n]$.

the $\frac{M}{2}$ point, which always coincides with one point of $h[n]$ (the central point of the sequence of $M+1$ points).

In order to see how this windowing process affects the frequency response of the filter (compared to its desired behaviour) the Modulation or Windowing Theorem is of convenient use:

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, W(e^{j(\omega-\theta)})\, d\theta \tag{11}$$

Note that Eq. (11) is a perdiodic convolution, i.e. a convolution of two periodic functions with the limits of integration extending over only one period. Overall, $H(e^{j\omega})$ is a "smeared" version of the desired filter response $H_d(e^{j\omega})$. A graphical picture of this effect is shown in [Oppenheim and Schafer, 2009]. Bear in mind that the window function $W(e^{j\omega})$ does not need to be rectangular (in fact, the rectangular window is the simplest). More enhanced window functions yield better approximations, also see [Oppenheim and Schafer, 2009] for further details.

In the end, the obtained filter $H(e^{j\omega})$ has a frequency response like

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{-j\omega\frac{M}{2}} \tag{12}$$

In order to measure the linearity of the phase, the group function $\tau(\omega)$ represents a convenient measure:

$$\tau(\omega) = grd[H(e^{j\omega})] = -\frac{d}{d\omega}\angle H(e^{j\omega}) \tag{13}$$

Therefore, since $\tau(\omega) = \frac{M}{2}$, i.e. a constant delay for all frequencies, $H(e^{j\omega})$ is a linear-phase system and thus no dispersion in time of the output signal energy is present (no phase distortion).

References

[Oppenheim and Schafer, 2009] Oppenheim, A. V. and Schafer, R. W. (2009). *Digital Signal Processing*. Prentice–Hall, third edition.