Multirate Discrete-Time Signal Processing

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2010

Abstract

This tutorial presents a thorough description of procedure to change the sampling rate using discrete-time processing.

1 Introduction

The sampling premise states that a continuous-time signal $x_c(t)$ can be represented by a discrete-time signal $x[n]$, that consists of a sequence of samples

$$x[n] = x_c(nT) \tag{1}$$

related by the sampling period T. Then, it is possible to change the sampling rate of the discrete-time signal, i.e., to obtain a new discrete-time representation of the underlying continuous-time signal of the form

$$x_1[n] = x_c(nT_1) \tag{2}$$

where $T_1 \neq T$. This operation is called *resampling*. Conceptually, $x_1[n]$ might be obtained from $x[n]$ by reconstructing $x_c(t)$ and the resampling it with the new sampling period T_1. However, this approach is not practical due to the nonidealities of the D/A and A/D procedures. Thus, it is of interest to consider methods of changing the sampling rate that involve only discrete-time operations.

The process of reducing the sampling rate is called *downsampling*, while the inverse process is called *upsampling*. The sampling rate is changed in order to increase the efficiency of various signal processing operations. By doing so, the requirements of the filters may be relaxed and thus their complexity may be reduced significantly.

This tutorial is mostly based on [Oppenheim and Schafer, 2009]. Refer to it for further details.

2 Downsampling by a natural factor

The downsampling process implies that $T_1 > T$ regarding Eq. (3). Therefore, the new sequence results

$$x_d[n] = x[nM] = x_c(nT_d) = x_c(nMT) \tag{3}$$

It follows that the $x_d[n]$ is the sequence that would be obtained from $x_c(t)$ by sampling with period $T_d = MT$, where $M \in \mathbb{N}$ and strictly $M > 1$. From this operation it results that the bandwidth of $x_d[n]$ widens a factor of M wrt the bandwidth of $x[n]$, in addition to a magnitude attenuation of $\frac{1}{M}$. Hence, as long as the maximum frequency component of $x[n]$ remains lower than $\frac{\pi}{M}$, no aliasing is produced. In order to avoid any aliasing distortion problem, though, $x[n]$ should be prefiltered with a lowpass filter (in the discrete-time domain) with $\omega_c = \frac{\pi}{M}$. With this prefiltering scenario, the process is called *decimation*, see Figure 1.

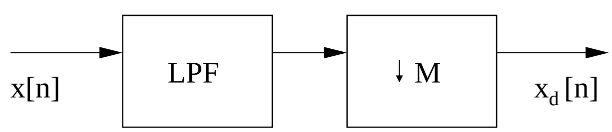

Figure 1: Decimation process: an antialiasing filter precedes the sampling rate compressor.

The frequency domain relation between the input and the output of the compressor is obtained with the discrete-time Fourier transform (DTFT) of the signals involved. For an input signal $x[n] = x_c(nT)$, the DTFT of this sequence is related to the Fourier Transform of the continuous-time input signal by $\omega = \Omega T$, therefore

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right] \tag{4}$$

Similarly, the DTFT of $x_d[n] = x[nM] = x_c(nMT)$ is

$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{MT} - \frac{2\pi r}{MT} \right) \right] \tag{5}$$

Note that $X_d(e^{j\omega})$ is attenuated by a factor of M wrt $X(e^{j\omega})$ and spread in frequency by the same factor, see Figure 2.

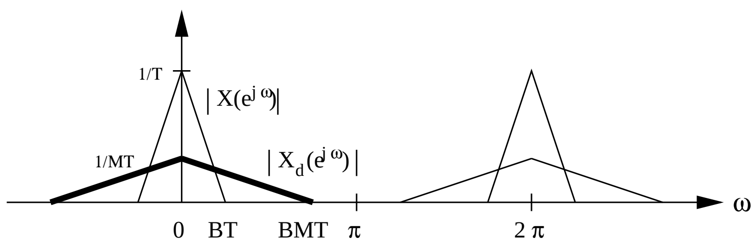

Figure 2: Frequency representation of the downsampling process.

Eq. (5) is related with Eq. (4) by the summation index r, as it can be expressed as

$$r = i + kM \tag{6}$$

where k and i are integers such that $-\infty < k < \infty$ and $0 \leq i \leq M-1$. As a result, Eq. (7) expresses the Fourier transform of the discrete-time sequence $x_d[n]$ (with sampling period M) in terms of the Fourier transform of the sequence $x[n]$.

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)} \right) \tag{7}$$

Another interpretation of Eq. (7) would consider the ideal sampling, with sampling period M, of the discrete-time sequence $x[n]$. Given that the discrete-time modulating sampling sequence $s[n]$ is periodic with period M, its discrete Fourier series (DFS) representation requires M harmonically-related complex exponentials

$$s[n] = \sum_{m=-\infty}^{\infty} \delta[n - mM] \xrightarrow{DFS} s[n] = \frac{1}{M} \sum_{i=0}^{M-1} e^{j\frac{2\pi i}{M}n} \tag{8}$$

Note that the DTFT of $x_d[n] = v[nM]$, where $v[n] = x[n] \cdot s[n]$, leads to the same result as in Eq. (7):

$$
\begin{aligned}
V(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} v[n]\, e^{-j\omega n} && (9)\\
&= \frac{1}{M} \sum_{i=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\left(\omega - \frac{2\pi i}{M}\right) n} && (10)\\
&= \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\omega - \frac{2\pi i}{M}\right)}\right) && (11)
\end{aligned}
$$

$$
\begin{aligned}
X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_d[n]\, e^{-j\omega n} && (12)\\
&= \sum_{n=-\infty}^{\infty} v[nM]\, e^{-j\omega n} \;\leftarrow\; m = nM && (13)\\
&= \sum_{m=-\infty}^{\infty} v[m]\, e^{-j\frac{\omega}{M} m} = V\left(e^{j\frac{\omega}{M}}\right) && (14)\\
&= \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi i}{M}\right)}\right) && (15)
\end{aligned}
$$

3 Upsampling by a natural factor

Consider a signal $x[n]$ whose sampling rate is to be increased by a factor of L, being $L \in \mathbb{N}$. Considering the underlying continuous-time signal $x_c(t)$, the objective is to obtain samples

$$x_i[n] = x_c(nT_i) \tag{16}$$

where $T_i = \frac{T}{L}$, from the sequence of samples $x[n] = x_c(nT)$, hence using only discrete-time processing tools. The operation of increasing the sampling rate is called *upsampling*. It follows that

$$x_i[n] = x[n/L] = x_c(nT/L) \qquad n = 0, \pm L, \pm 2L, ... \tag{17}$$

Graphically,

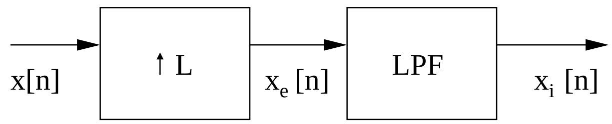

Figure 3: Interpolation process: a reconstruction filter follows the sampling rate expander.

The input discrete-time signal $x[n]$ is first processed through an expander, obtaining $x_e[n]$,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL] \tag{18}$$

Then, a lowpass filter with gain L and $\omega = \frac{\pi}{L}$ follows to reconstruct the modulated discrete-time impulse train sequence $x_e[n]$.

In the frequency domain

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_e[n]\,e^{-j\omega n} &(19)\\ &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]\right) e^{-j\omega n} &(20)\\ &= \sum_{k=-\infty}^{\infty} x[k]\,e^{-j\omega Lk} &(21)\\ &= X(e^{j\omega L}) &(22)\end{aligned}$$

Therefore, $X_e(e^{j\omega})$ is a frequency-scaled version of $X(e^{j\omega})$, so that ω is normalised by

$$\omega = \Omega\,T_i = \Omega\,\frac{T}{L} \tag{23}$$

see Figure 4.

Note that with this formulation the amplitude of the output of the interpolating filter is $\frac{L}{T} = \frac{1}{T_i}$, that is the same that would be obtained from sampling $x_c(t)$ with a sampling period of T_i directly.

This output satisfies Eq. (16) if the input sequence $x[n]$ was obtained by sampling without aliasing. Therefore, the system implementing this process is called an *interpolator*, since it fills the missing samples, and the operation of upsampling is consequently considered to be synonymous with interpolation.

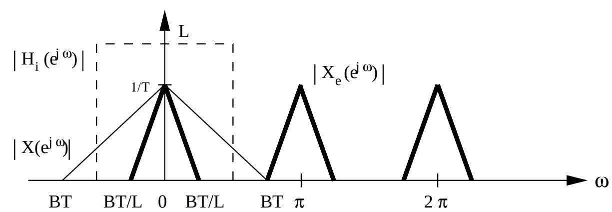

Figure 4: Frequency representation of the upsampling process.

For an ideal implementation of the interpolation filter, its output would be determined by its sinc-shaped impulse response

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \, \frac{sin[\pi(n - kL)/L}{\pi(n - kL)/L} \tag{24}$$

3.1 Linear interpolation filter

Although ideal lowpass filters for interpolation cannot be implemented, their behaviour can be well approximated with a simple linear interpolation procedure. Linear interpolation provides the samples between two original samples (that would be obtained if a higher sampling frequency was used) in a straight line joining the two original sample values. In discrete-time signal processing, this behaviour may be attained with a triangularly shaped impulse response filter, see Eq. (25), applied on the expanded sequence of samples (obtained by inserting zeros).

$$h_{lin}[n] = \begin{cases} 1 - \frac{|n-L|}{L}, & |n-L| < L \\ 0, & otherwise \end{cases} \tag{25}$$

4 Sampling rate conversion by a rational factor

By combining interpolation and decimation in cascade (and in this order), it is possible to change the sampling rate by a rational factor. This combination produces an output sequence $\tilde{x}_d[n]$ that has an effective sampling period of TM/L.

If $M > L$ there is a net increase in the sampling period (a decrease in the sampling rate), and if $M < L$, the opposite is true. For practical purposes, the filters in question are combined as is shown in Figure 5.

According to the relation of M wrt L, either one or the other cutoff frequency is the most dominant. So, the minimum of the two is taken for the combination.

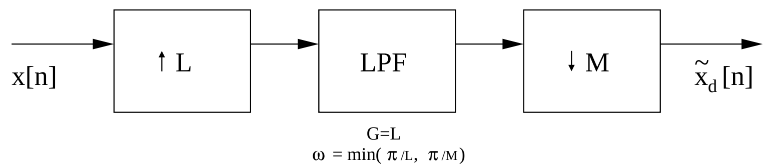

Figure 5: Interpolation and decimation procedures in cascade.

5 Applications

The goal of modifying the sampling rate is to increase the efficiency of various signal processing operations. A couple of situations where this objective is exemplified are:

- Upsampling before D/A conversion in order to relax the requirements of the analog lowpass antialiasing filter. This technique is used in audio CD, where the sampling frequency 44.1kHz is increased fourfold to 176.4kHz before D/A.
- In speech processing, a data compression may be applied by decomposing the speech signal into several components, which may be quantised with different word lengths.

References

[Oppenheim and Schafer, 2009] Oppenheim, A. V. and Schafer, R. W. (2009). *Digital Signal Processing.* Prentice–Hall, third edition.